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Instability of ultra-spinning black holes

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Abstract: It has long been known that, in higher-dimensional general relativity, there are black hole solutions with an arbitrarily large angular momentum for a fixed mass [1]. We examine the geometry of the event horizon of such ultra-spinning black holes and argue that these solutions become unstable at large enough rotation. Hence we find that higher-dimensional general relativity imposes an effective "Kerr-bound" on spinning black holes through a dynamical decay mechanism. Our results also give indications of the existence of new stationary black holes with "rippled" horizons of spherical topology. We consider various scenarios for the possible decay of ultra-spinning black holes, and finally discuss the implications of our results for black holes in braneworld scenarios.

Keywords: Extra Large Dimensions, Classical Theories of Gravity, Black Holes.

Contents

1. Introduction

General relativity in higher dimensions is an active area of ongoing studies in both string theory and particle theory. In particular, investigations over the past fifteen years have produced an impressive catalogue of black hole solutions for various effective theories of Einstein gravity coupled to many different kinds of matter fields [2]. However, there have long been indications that the physics of event horizons in higher-dimensional general relativity is far richer and complex than in the standard four-dimensional theory.

A simple example illustrating this point is the theorem forbidding "topological hair" for four-dimensional black holes. That is, each connected component of a stationary event horizon must have the topology of a two-sphere.[1] Of course, this result is easily evaded in higher dimensions. As an example, consider the four-dimensional Schwarzschild metric combined with a flat metric on $\mathbb{R}^n$. This space-time is an extended black "brane" solution of Einstein's equations in $4{+}n$ dimensions, and the topology of the horizon is $S^2 \times \mathbb{R}^n$. Recently, an even more dramatic violation was presented in ref. [4]. There, an asymptotically flat solution was constructed describing a rotating black ring in five dimensions, with horizon topology $S^2 \times S^1$. In fact for certain values of mass and spin, one finds that there exist *three* solutions, a black hole with an S^3 horizon and two different black rings

[1] See proposition 9.3.2 in [3].

with an $S^2 \times S^1$ horizon. This is a clear indication that, unlike four dimensions, in higher dimensions black hole solutions will not be completely determined by a few asymptotic charges (such as the mass and angular momentum).

Further, in contrast to the stability proven for four-dimensional black holes [5], Gregory and Laflamme made the surprising discovery that the extended black branes appearing in higher dimensions are unstable [6]. Ten years later, the question of the endpoint reached with the onset of this instability remains unresolved [7]. However, this question lead to a conjecture that inhomogeneous black brane solutions may also exist [8]. This recent conjecture produced a surge of activity in this area [9], including the discovery of, at least, a certain class of such solutions [10]. Hence, while in four dimensions black hole solutions are extremely constrained by the uniqueness theorems, such results do not appear to apply for higher dimensions. Rather, it seems that we currently have only seen glimpses of a rich landscape of solutions with vast unexplored areas.

In this paper, we present a preliminary study of the stability of rotating black holes in higher dimensions. In particular, it was discovered over fifteen years ago that Einstein's equations in higher dimensions have solutions describing black holes with an arbitrarily large angular momentum for a fixed mass [1]. We refer to such solutions as "ultra-spinning" black holes. No such black holes arise in four-dimensional relativity where all solutions satisfy the famous Kerr bound, $J \leq GM^2$. However, we will argue that the ultra-spinning black holes are in fact unstable and so an effective Kerr bound seems to arise in higher dimensions through a dynamical decay mechanism.

Since the analytic theory of the perturbations of higher-dimensional rotating black holes has not been fully developed yet, our approach is heuristic. We anticipate that our conclusions will be useful in guiding future research in the area. In particular, we have identified a limit (of infinite rotation) for these black holes where an instability is known to occur, and therefore it would make sense to focus on the subsector of the perturbations where these unstable modes are expected.

Another consequence of our study is an indication of the existence of new rotating black holes, of spherical topology, where the horizon is distorted by ripples along the polar direction in a way that preserves the same symmetries as the smoother black holes already known.

The remainder of this paper is organized as follows: section 2 reviews the metric for a rotating black hole in d dimensions with one nonvanishing spin parameter. The focus of the following discussion will be $d \geq 6$. In section 3, we examine the geometry of the event horizon of this solution when the spin becomes large. We show that the horizon takes the shape of a (higher-dimensional) "pancake", spreading out in the plane of rotation while becoming narrowly contracted in the other spatial directions. Section 4 presents a limit in which this black hole geometry becomes an unstable black membrane extending over the original plane of rotation. We argue that the instability appearing in this limit must also be relevant in the ultra-spinning regime. In section 5 we briefly comment on the thermodynamics of ultra-spinning black holes. Section 6 considers the possible decay of the ultra-spinning black hole by fragmentation into multiple black holes. Finally, we conclude in section 7 with a discussion of the implications of our result. While

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throughout the main text we focus on higher-dimensional black holes rotating in a single plane, this is only to simplify the presentation. In the appendix A, we demonstrate how an analogous black brane limit can be taken for black holes rotating in multiple planes and argue that hence instabilities will also arise in cases when several spin parameters become large simultaneously. Appendix B presents some calculations related to the possible decay of ultra-spinning black holes by the emission of gravitational radiation.

2. Ultra-spinning black holes

To begin our considerations of spinning black holes in higher dimensions, we recall the description of angular momentum in higher dimensions. In d dimensions, the rotation group is $SO(d-1)$ for which there are[2] $\lfloor(d-1)/2\rfloor$ distinct Casimirs which could be used to characterize the angular momentum of a system. More concretely, the latter is described by an angular momentum two-form. By going to the center-of-mass frame and then making a suitable rotation of the spatial coordinates, this two-form can be put in a block-diagonal form containing $\lfloor(d-1)/2\rfloor$ parameters J_i. Each of the J_i corresponds to the angular momentum associated with motion or rotation in distinct orthogonal (spatial) planes of the higher-dimensional spacetime.

Hence, a black hole in d dimensions is in general characterized by $\lfloor(d-1)/2\rfloor$ J_i, corresponding to spins in orthogonal planes. While in four dimensions this counting yields the usual single J, for $d \geq 5$ the general solution contains two or more spins. This prolonged review simply sets the stage for our statement that, throughout the following, we will focus our discussion to higher-dimensional black holes rotating in a single plane. (Further, for physical reasons that will become apparent below, the discussion will be restricted to $d \geq 6$.) However, we stress that this restriction is made only to simplify the presentation. In appendix A, we will extend the discussion to black holes with multiple spins and in particular we demonstrate how analogous instabilities can arise in this case.

Myers and Perry found general solutions describing asymptotically flat black holes in d dimensions with all $\lfloor(d-1)/2\rfloor$ J_i nonvanishing [1]. In the case with a single non-zero spin $J_1 = J \neq 0$ (and $J_{i>1} = 0$), the metric reduces to

$$\begin{aligned} ds^2 = & -dt^2 + \frac{\mu}{r^{d-5}\rho^2}\left(dt + a\sin^2\theta\, d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta\, d\varphi^2 + \\ & + r^2\cos^2\theta\, d\Omega^2_{(d-4)}\,, \end{aligned} \tag{2.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 + a^2 - \frac{\mu}{r^{d-5}}\,. \tag{2.2}$$

The physical mass and angular momentum are related to the parameters μ and a by

$$M = \frac{(d-2)\Omega_{d-2}}{16\pi G}\mu\,, \qquad J = \frac{2}{d-2}Ma\,, \tag{2.3}$$

where Ω_{d-2} is the area of a unit $(d-2)$-sphere. Hence one can think of a as essentially the angular momentum per unit mass.

[2] We use the notation $\lfloor s \rfloor$ to denote the integer part of s.

The first line in eq. (2.1) looks very much like the metric of its four-dimensional counterpart, the Kerr solution, with the $1/r$ fall-off replaced, in appropriate places, by $1/r^{d-3}$ — of course, with the choice $d = 4$ this is precisely the Kerr metric. The second line contains the line element on a $(d-4)$-sphere which accounts for the additional spatial dimensions. Heuristically, we can see the competition between gravitational attraction and centrifugal repulsion by considering the expression

$$\frac{\Delta}{r^2} - 1 = -\frac{\mu}{r^{d-3}} + \frac{a^2}{r^2}\,. \tag{2.4}$$

Roughly, the first term on the right-hand side corresponds to the attractive gravitational potential and falls off in a dimension-dependent fashion. In contrast, the repulsive centrifugal barrier described by the second term does not depend on the total number of dimensions, since rotations always refer to motions in a plane. As a consequence, the features of the event horizons will be strongly dimension-dependent.

The outer event horizon can be determined as the largest (real) root r_+ of $g_{rr}^{-1} = 0$ or simply $\Delta(r) = 0$. That is,

$$r_+^2 + a^2 - \frac{\mu}{r_+^{d-5}} = 0\,. \tag{2.5}$$

For $d = 4$, a regular horizon is present for values of the spin parameter a up to the Kerr bound: $a = \mu/2$ (or $a = GM$), which corresponds to an extremal black hole with a single degenerate horizon (with vanishing surface gravity). Solutions with $a > GM$ correspond to naked singularities. In $d = 5$, the situation is apparently quite similar since the real root at $r_+ = \sqrt{\mu - a^2}$ exists only up to the extremal limit $\mu = a^2$. However, this extremal solution has zero area, and in fact, it is a naked ring singularity. Note that the five-dimensional black ring of ref. [4] evades this bound. We return to a brief discussion of five dimensions in section 7.

For $d \geq 6$, $\Delta(r)$ is always positive at large values of r, but the term $-\mu/r^{d-5}$ makes it negative at small r. Therefore Δ always has a (single) positive real root independent of the value of a. Hence regular black hole solutions exist with arbitrarily large a, i.e., arbitrarily large angular momentum per unit mass. We refer to such solutions with large angular momentum per unit mass as "ultra-spinning" black holes. In the general situation with multiple spin parameters, an ultra-spinning black hole would be one for which any of a_i satisfies $a_i^{d-3} > \mu$. As will be discussed in appendix A, it is certainly possible to have a regular horizon where several of the spin parameters satisfy this inequality.

3. Geometry of ultra-spinning horizons

Now we consider the shape of the event horizon of an ultra-spinning black hole. To do so, we compare the sizes of invariant geometric quantities such as the proper areas of different sections of the horizon.

In the limit of very large a and fixed mass, the coordinate radius of the horizon is well approximated by

$$r_+ \simeq \left(\frac{\mu}{a^2}\right)^{1/(d-5)} \ll a\,. \tag{3.1}$$

Observe that r_+ shrinks as a is increased keeping the mass fixed. However, r_+ by itself does not have any invariant meaning.

The total, $(d-2)$-dimensional area of the horizon is

$$\mathcal{A} = \Omega_{d-2} r_+^{d-4}(r_+^2 + a^2) \simeq \Omega_{d-2} r_+^{d-4} a^2 \simeq \Omega_{d-2}\left(\frac{\mu^{d-4}}{a^2}\right)^{1/(d-5)}\,. \tag{3.2}$$

Note that this area *decreases* as angular momentum increases with a fixed mass. This is also the case in $d=4$, but now for higher d we can consider the limit $a \to \infty$ and we see that the area shrinks to zero in the limit.

Now consider the area of a two-dimensional section of the horizon obtained by considering the solution (2.1) at a fixed point in the "transverse" sphere Ω_{d-4}. This "parallel" two-dimensional area is

$$\mathcal{A}_{||}^{(2)} = \Omega_2 (r_+^2 + a^2) \simeq \Omega_2 a^2\,. \tag{3.3}$$

This area *grows* for ultra-spinning black holes. Alternatively, one might fix the angles θ and φ to consider the area of the transverse sphere with the result

$$\mathcal{A}_{\perp}^{(d-4)} = \Omega_{d-4}(r_+ \cos\theta)^{d-4}\,. \tag{3.4}$$

These observations are easily compatible if we consider that the horizon has a characteristic size $\ell_{||}$ in directions parallel to the rotation plane, and $\ell_\perp$ in directions perpendicular to it. Then, if $\ell_{||} \gg \ell_\perp$, we expect that

$$\begin{aligned} \mathcal{A} &\propto \ell_{||}^2\, \ell_\perp^{d-4} + \cdots \\ \mathcal{A}_{||}^{(2)} &\propto \ell_{||}^2 + \cdots \\ \mathcal{A}_{\perp}^{(d-4)} &\propto \ell_\perp^{d-4} + \cdots\,, \end{aligned} \tag{3.5}$$

where the dots denote terms small in an expansion in $\ell_\perp/\ell_{||}$. Comparing with eqs. (3.2) to (3.4) above, we easily identify

$$\begin{aligned} \ell_{||} &\sim a\,, \\ \ell_\perp &\sim r_+\,. \end{aligned} \tag{3.6}$$

This result is further supported by examining the capture of null geodesics in the plane of rotation — see appendix B.1. So ultra-spinning black holes are characterized by very large $\ell_{||}$ and very small $\ell_\perp$: the horizon is highly "pancaked" along the plane of rotation. That is, in accord with the naive intuition, the horizon of these rapidly rotating black holes spreads out in the plane of rotation while contracting in the transverse directions.

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4. Black membrane limit

We now introduce a limit where the rotation parameter a grows to infinity for the metric (2.1) with $d \geq 6$. If we keep the mass (i.e., μ) finite as $a \to \infty$, then the horizon radius (3.1) and area (3.2) shrink to zero. Since we are interested in keeping a non-vanishing horizon, we will let $\mu \to \infty$ with $\hat{\mu} \equiv \mu/a^2$ held fixed.

In this limit, if we are near the horizon away from the pole $\theta = 0$, the rotation grows without bound. So in order to produce a finite metric, we will confine ourselves to progressively smaller regions near $\theta = 0$. It is convenient to introduce a new coordinate $\sigma = a \sin\theta$ which is kept finite as $a \to \infty$ and therefore focuses on $\theta \to 0$. We also keep r finite. Taking this limit for the solution (2.1) we obtain

$$ds^2 = -\left(1 - \frac{\hat{\mu}}{r^{d-5}}\right) dt^2 + \frac{dr^2}{1 - \frac{\hat{\mu}}{r^{d-5}}} + r^2 d\Omega^2_{(d-4)} + d\sigma^2 + \sigma^2 d\varphi^2 \,. \tag{4.1}$$

This is the metric of a black membrane in d dimensions, extended along the plane (σ, φ). The tension of the membrane is proportional to $\hat{\mu} = \mu/a^2$ and so is given by the limiting mass per unit area of the original black hole. The horizon in eq. (2.1) began with the topology of a sphere S^{d-2} and has now pancaked out in the plane of rotation producing the topology $\mathbb{R}^2 \times S^{d-4}$ in the limiting metric (4.1). The angular momentum is not longer visible in this limit since

$$g_{t\varphi} \to \frac{\hat{\mu}\sigma}{r^{d-5}} \frac{\sigma}{a} \to 0 \,, \tag{4.2}$$

i.e., in this limit with $\sigma \ll a$, the rotation of the horizon becomes negligible.

The existence of this limit has a remarkable consequence. Gregory and Laflamme (GL) have shown that black branes are classically unstable [6]. Therefore, in the limit of infinite rotation six- or higher-dimensional black holes go over to an unstable configuration. It is then natural to conjecture that the instability sets in already at finite values of the rotation, so ultra-spinning black holes become classically unstable for large enough a.

Let us now argue in more detail that the instability must be present at large but finite rotation, and is not simply an artifact of the the limit taken above. When a is large but finite, the geometry near the horizon and near the axis of the ultra-spinning black hole is well approximated by the metric (4.1) for $r, \sigma < a$. More precisely, one finds that the diagonal components of the metric are as given above in eq. (4.1) with corrections which are suppressed by factors of r^2/a^2 or σ^2/a^2. Similarly, as indicated above, the only off-diagonal component $g_{t\varphi}$ is suppressed by σ/a. Hence the spacetime geometry near the center of the "pancake" horizon is essentially that of the black membrane (4.1).

The wavelength of the threshold mode of the GL instability is $\lambda_{GL} \sim \hat{\mu}^{1/(d-5)} \simeq r_+$. Since in the ultra-spinning regime $r_+ \ll a$, or $\mu \ll a^{d-3}$, there is ample room on the horizon to accommodate unstable fluctuations. In particular, one can construct initial data for localized wave packets containing unstable modes with support in a finite region near the horizon of the black membrane (4.1). It would then be possible to import this initial data with only minor perturbations to the pole region of the horizon of an ultra-spinning black hole, and in this setting the future evolution should display the same growth of unstable modes as for the full black membrane. In any event, it seems unlikely that the boundary conditions at the edge of the ultra-spinning horizon could eliminate the unstable modes, since the edge can be made to lie arbitrarily far away from the pole.

In appendix A, we extend the above discussion to black holes with several rotation parameters turned on. If at least one (in even d) or two (in odd d) of the spins are much smaller than the rest, then these black holes can have an ultra-spinning regime where the horizon is flattened along each of the ultra-spinning rotation planes. For example, if n spins grow to infinity (with $n < \lfloor (d-3)/2 \rfloor$ and $d \geq 6$), one obtains, with the appropriate limit, a black $2n$-brane solution.

5. Remarks on thermodynamics

It has long been supposed that the classical GL instability has a connection with the thermodynamic properties of the black branes. In their original work [6], GL gave an argument based on global thermodynamic stability to suggest that the unstable black string would fragment into a chain of localized black holes. While this scenario is controversial [8], sharper conjectures as the relation between the classical and local thermodynamic stability of black branes have been recently formulated [11]. In these discussions, the appearance of a negative specific heat of a black brane is related to the onset of a classical instability. Previous investigations have focused on charged black branes [12] and little consideration has been given to spin [13]. In this direction, let us comment that for the Kerr black hole, the phase transition signaled by the divergence of the specific heat (at constant J) [14] is not associated to any classical instability of the Kerr black hole, which is believed to be stable. Instead, following [11], one would associate this phase transition with a transition towards classical stabilization of the Kerr black string at a finite spin. Further work in this direction is in progress.

Before leaving this topic, however, we observe that the thermodynamics of the spinning black holes (2.1) show a qualitative change in behavior. That is, these black holes make a transition from behaving similar to the Kerr black hole, to behaving like a black membrane. The simplest and perhaps clearest quantity to consider is the black hole temperature

$$T = \frac{1}{4\pi}\left(\frac{2r_+^{d-4}}{\mu} + \frac{d-5}{r_+}\right). \tag{5.1}$$

For fixed mass and increasing spin, r_+ always decreases. So, beginning from zero spin, the first term in the r.h.s. of (5.1) dominates and T always decreases, like in the familiar case of the Kerr black hole. In $d = 4$ and $d = 5$ the temperature shrinks to zero at extremality, but in $d \geq 6$ there is no extremal limit. Instead, T reaches a minimum and then starts growing like $\sim r_+^{-1}$, as expected for the black membrane. The minimum, where the behavior changes, can be determined exactly as

$$\frac{a}{r_+} = \sqrt{\frac{d-3}{d-5}}\,. \tag{5.2}$$

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For the dimensionless quantity a^{d-3}/μ one finds the critical values

$$\frac{a^{d-3}}{\mu} \simeq 1.29 \qquad (d=6)\,; \qquad 1.33 \qquad (d=7)\,; \qquad 1.34 \qquad (d=8)\,. \tag{5.3}$$

This analysis seems to indicate that the membrane-like behavior, and hence the instability, occurs at a relatively low value of the angular momentum, i.e., not too far into the ultra-spinning regime. In the next section we present further evidence for this.

6. Death by fragmentation

The arguments of the previous sections provide a clear indication that ultra-spinning black holes become classically unstable as the rotation parameter grows. A precise determination of the angular momentum per unit mass at which the instability appears requires a linearized analysis of the perturbations of the Myers-Perry black holes (which may show unstable modes other than the ones suggested by the limiting GL instability). This is a difficult but important task that we will not pursue in this paper, where we remain at a more heuristic level.

Furthermore, the linearized perturbation analysis does not reveal by itself what is the final fate of the black hole. Indeed, the fate of black branes undergoing the GL instability has been a matter of debate over the last few years. As we mentioned in the previous section, GL gave a thermodynamic argument for the fragmentation of a black string [6] and we return to this reasoning here. The essential idea was that a black string compactified on a large enough circle could increase the horizon area if it pinched off to become a black hole (of the same mass) localized in the circle. Hence such a process would be driven by the second law. Further, this thermodynamic transition happens when the compact circle can also fit the wavelength of GL-unstable modes, suggesting a connection between both instabilities. So [6] proposed that the rippling of the horizon observed at linearized order grows until the horizon pinches off down to planckian-size necks. At this point, quantum gravity effects should split off the black string into black holes. After ref. [8], this picture has become more controversial. Nevertheless, it is remarkable that the global thermodynamical argument appears to be linked quite precisely to the classical instability. Certainly we are not aware of any counterexamples to this connection. Therefore in the following we will consider the instability of the ultra-spinning higher-dimensional black holes from the simple point of view of the entropy arguments presented in ref. [6]. That is, we wish to compare the horizon area of an ultra-spinning black hole to that of a plausible final state. Whenever the total final horizon area is larger than the area of the initial black hole, we take it as an indication of an instability.

We examine the possibility that the ultra-spinning black hole breaks apart into two identical black holes carrying away the spin as orbital angular momentum. This could happen if, e.g., the horizon develops a 'bar-mode' instability which eventually disrupts the horizon. Our approximation is rather crude since we neglect the emission of gravitational radiation along the process, but this is the same as in the original GL argument. We also take the final black holes to be non-spinning, since this maximizes the final entropy. For the same reason, we do not describe the fragmentation into more than two black holes. We take the final black holes to be of equal mass; it is easy to allow for different masses, but it leads to essentially the same results.

The initial state is characterized by the black hole mass M and spin J, or the parameters μ and a in (2.3). Recall that the initial area (3.2) is

$$\mathcal{A}_0 = \Omega_{d-2} r_+^{d-4}(r_+^2 + a^2)\,. \tag{6.1}$$

The final state consists of two black holes, each of mass m, infinitely far from each other, and moving in antiparallel directions, with impact parameter $2R$. The latter will be fixed later on. If the total energy and angular momentum are M and J, then, in the center-of-momentum frame, the momenta of the final black holes are $\pm J/2R$, so

$$M = 2\sqrt{m^2 + \frac{J^2}{4R^2}}\,. \tag{6.2}$$

One can solve for m as

$$m = \frac{1}{2}\sqrt{M^2 - \frac{J^2}{R^2}}\,. \tag{6.3}$$

Defining

$$\mu_1 = \frac{16\pi G}{(d-2)\Omega_{d-2}} m \tag{6.4}$$

then (6.3) becomes

$$\mu_1 = \frac{r_+^{d-5}(r_+^2 + a^2)}{2}\sqrt{1 - \frac{4}{(d-2)^2}\frac{a^2}{R^2}}\,. \tag{6.5}$$

The final black holes are assumed to be non-rotating. They are moving apart at asymptotically constant velocity, but the black hole area is invariant under boosts [15]. So the total final area is

$$\mathcal{A}_1 = 2\Omega_{d-2}\mu_1^{\frac{d-2}{d-3}}\,. \tag{6.6}$$

Our criterion for an instability

$$\frac{\mathcal{A}_1}{\mathcal{A}_0} > 1 \tag{6.7}$$

becomes

$$\frac{1 + (a/r_+)^2}{2}\left(1 - \frac{4}{(d-2)^2}\frac{a^2}{R^2}\right)^{\frac{d-2}{2}} > 1\,. \tag{6.8}$$

It is clear that $\mathcal{A}_1$ will be maximized by setting R as large as possible, i.e., when the escaping black holes carry away the angular momentum with minimal kinetic energy and hence maximal rest mass and maximal horizon area. However, in the absence of further knowledge about the fragmentation process, it would not be reasonable to assume that R is much larger than the radius of the initial horizon in the rotation plane. For large rotation, we know that this radius is approximately equal to a. From (6.8) we see that if R scales as a then the inequality will always be satisfied for large enough a. In fact, this will be the case whenever R grows faster than $\frac{2}{d-2}a$. So in the ultra-spinning regime the entropic arguments seem to allow for a rather wide margin for fragmentation into two black holes.

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To obtain an estimate of the transition point, we make a specific choice requiring that R is not larger than the horizon radius,

$$R \leq \sqrt{r_+^2 + a^2}\,. \tag{6.9}$$

Saturating this inequality yields the following values for eq. (6.8) to be satisfied:

$$\frac{a}{r_+} \geq 1.36 \qquad (d=6)\,, \qquad 1.26 \qquad (d=7)\,, \qquad 1.20 \qquad (d=8)\,, \tag{6.10}$$

or equivalently,

$$\frac{a^{d-3}}{\mu} \geq 0.88 \quad (d=6)\,, \qquad 0.97 \quad (d=7)\,, \qquad 1.02 \quad (d=8)\,. \tag{6.11}$$

Alternative choices such as $R = k\sqrt{r_+^2 + a^2}$, or $R = ka$, with $k \leq 1$, lead to minor differences in the critical value of a/r_+ as long as $k > 2/(d-2)$ i.e., R scales faster than $\frac{2}{d-2}a$. The quantitative differences for different values of d are also small. Obviously, these values should not be taken too literally. The important result is that the area criterion (6.7) seems to signal an instability for values of a/r_+ not much larger than one.

We have also considered a variety of other possible final configurations, e.g., final state black holes with different masses, or two black holes flying apart, with a third one (rotating or not) remaining at the initial position. In all cases the qualitative results remain as above, with the critical value of a being always a few times r_+.

Of course, an alternative conjecture as to the evolution of the instability is that the ultra-spinning black holes decay by the emission of gravitational radiation. We comment on this scenario below in the next section and in appendix B.

7. Discussion

We have shown that in the ultra-spinning regime of a black hole, a wide portion of its horizon is well approximated by a flat black membrane. Taken together with the GL instability of black branes, we come to the conclusion that ultra-spinning black holes must be classically unstable. We have also shown that thermodynamic arguments, based on the second law, seem to support this view. Thus we find a dynamical "Kerr bound" on the spin in d dimensions of the form

$$J^{d-3} \leq \beta_d\, G M^{d-2}\,, \tag{7.1}$$

where the numerical factor

$$\beta_d = \frac{2^{d+1}\pi}{(d-2)^{d-2}\Omega_{d-2}} \left(\frac{a^{d-3}}{\mu}\right)_{\rm crit} \tag{7.2}$$

is fixed in $d \geq 6$ by the onset of an instability at critical values of the parameters. The simple estimates of sections 4 and 6 suggest that $(a^{d-3}/\mu)_{\rm crit}$ is not much larger than a few, with a weak dependence on d. Note that this implies that the numerical value of β_d becomes small rapidly with increasing d (recall $\beta_4 = 1$).

Throughout the preceding, we have only considered higher-dimensional black holes rotating in a single plane. However, we stress again that this restriction was made only to simplify the presentation. Appendix A presents an extension of the black brane limit in section 4 to black holes with multiple spins. Hence we argue that analogous instabilities arise for this case.

In section 6, evidence suggesting that $(a^{d-3}/\mu)_{\text{crit}} = O(1)$ came from extending the thermodynamic arguments of ref. [6] and considering the possibility that an ultra-spinning black hole could fragment into several black holes carrying orbital angular momentum. A more conventional suggestion (which does not rely on quantum gravity effects) would be that the rotating black hole horizon becomes distorted, and since the black hole is rotating gravitational waves are emitted. This radiation could by itself cause a spin-down returning the black hole to the stable regime. However, a detailed description of such a spin-down (or the gravitational radiation produced in any decay process) is beyond the scope of this paper. Appendix B presents some calculations which take some tentative steps in this direction by considering processes in which the black hole is only slightly perturbed away from the Myers-Perry solutions (2.1). However, it seems that these putative decay processes are in conflict with the area theorem — they do not seem to produce radiation and an increase in the total area at the same time, so they are probably forbidden. The most likely outcome is that the instability must produce highly nonlinear distortions of the horizon in order for gravitational radiation to provide an effective mechanism to dissipate the angular momentum of ultra-spinning black holes.

An important application of the limiting membrane geometry is the identification of the sector of the perturbations of the spinning black hole where the instability is expected. For the limiting membrane eq. (4.1), GL found an unstable tensor perturbation, regular on the horizon, and preserving the $\mathrm{SO}(d-3)$ symmetry of the transverse sphere S^{d-4}, with components of the generic form

$$h^{\mu\nu} \sim e^{\Omega t} e^{i(k_1 z^1 + k_2 z^2)} h(r) \,, \tag{7.3}$$

where z^1, z^2 are cartesian coordinates along the planar membrane direction. They found that when $k = \sqrt{k_1^2 + k_2^2}$ is below a critical wavenumber k_c ($\sim r_+^{-1}$), then Ω becomes real and positive, hence the perturbation is unstable. At the threshold $k = k_c$, there is a static, zero-mode perturbation with $\Omega = 0$. Note that Ω is a function of k only, and not of k_1 or k_2 separately. Therefore, if instead of the plane wave basis in (z^1, z^2) in (7.3), we choose a cylindrical basis in polar coordinates (σ, φ) with Bessel wavefunctions,

$$h^{\mu\nu} \sim e^{\Omega t} J_m(k\sigma) e^{im\varphi} h(r) \,, \tag{7.4}$$

then the azimuthal number m is irrelevant for the GL perturbations, in the sense that whenever $k < k_c$, the instability will be present for any value of m. So, in particular, there do exist unstable modes in the axially symmetric sector $m = 0$. The radial profile of one such unstable mode is a cylindrical wave $J_0(k\sigma)$.

Importing these observations to the perturbation problem of ultra-spinning black holes, where we trade σ for the polar angle θ, we see that in order to identify an unstable mode we can restrict ourselves to perturbations that depend only on r and θ, and which preserve all the rotation symmetries $SO(2) \times SO(d-3)$ of the black hole. So the instability need not break the axial symmetry of the rotating black hole, and this unstable mode will not be radiated away, at least at this linearized order. Other modes which are not axially symmetric presumably become more complicated as one reaches the region of the horizon where rotation ceases to be negligible. Such perturbations should cause the black hole to emit gravitational waves, as considered above.

An interesting recent study of the stability problem focused on scalar perturbations of the five-dimensional rotating black hole, and also on axially symmetric scalar perturbations in higher dimensions.[3] No signs of instability were found [16]. This is indeed compatible with our analysis, since black branes are known to be stable to scalar perturbations [6]. Therefore the sector where the unstable mode indicated by the limiting GL instability is expected to lie, has not been probed yet. There is of course the possibility of unstable perturbations besides the ones we have identified.

Consider now the threshold, zero-mode axisymmetric perturbation with $k = k_c$. It has been shown in [10] that the GL zero-mode signals a new branch of black brane solutions, which break the translation invariance along the brane. These are inhomogeneous black branes, with an energy higher than the corresponding homogeneous branes. Their existence suggests the possibility of a similar new class of stationary rotating black holes with a rippled horizon of spherical topology. For one such ultra-spinning rippled black hole, the geometry near the pole $\theta \ll 1$ and $r \ll a$ should be well approximated by an axially symmetric inhomogeneous black membrane. The ripples have a profile in the polar angle θ, and preserve all the rotational $SO(2) \times SO(d-3)$ symmetry of the horizon. If one were able to develop at least the relevant part of the perturbation theory of spinning black holes, then the rippled rotating black holes should branch-off from axisymmetric zero-mode perturbations. Of course, these black holes may be unstable themselves, e.g., to perturbations that break axial symmetry.

It has been conjectured that there should exist spinning black hole solutions with only two Killing isometries [18]. However, the new solutions we are proposing are not of this sort, since they will have the same number of Killing isometries as the already known black holes. Even if we are conjecturing their existence from membrane solutions that break some symmetry, the radial symmetry that is broken is absent at any finite values of the rotation.

If such solutions actually exist, then it would seem impossible to have a version of black hole uniqueness within the class of black holes with a given topology of the horizon. Our study might, perhaps, be taken to add to the conjecture that local stability, which plays no role in the proof of uniqueness in four dimensions, may be the feature that selects, in higher dimensions, one solution among others with the same asymptotic charges [19]. However, the catalogue of solutions and their properties are too poorly understood at present to extract any conclusions in this respect.

[3]The same equations for scalar perturbations have been obtained by a number of other researchers [17]. The separability of the variables r and θ has been noted by all of them, but only ref. [16] have investigated stability in detail.

We have focused on $d \geq 6$, but the situation in $d = 5$ is also quite interesting. Although these black holes cannot be ultra-spinning, their geometry close to extremality is also a thin pancake of radius $\ell_{||} \sim a$ and thickness $\ell_\perp \sim r_+$. However, there is no black brane limit, since in $d = 5$ there are no black two-branes. So there is no robust argument for an instability in this case.

Nevertheless, it has been conjectured that an instability sets in before the extremal solution is reached [4]. We observe further that the argument for fragmentation in section 6 applies as well in $d = 5$. The extremal limit in $d = 5$ corresponds to $a^2 \to \mu$, $r_+ \to 0$, i.e., $a/r_+ \to \infty$. With the choice (6.9), the area criterion (6.8) implies the critical value $a/r_+ \simeq 1.60$, or

$$\left(\frac{a^2}{\mu}\right)_{\rm crit} \simeq 0.72 \qquad (d = 5)\,. \tag{7.5}$$

So even if this case has no black brane limit, an instability near extremality could be expected. Instead, in $d = 4$ we cannot have $a > r_+$ and, consistently, none of the previous arguments predict any instability.

On the other hand, the five-dimensional black ring of [4] does have an ultra-spinning regime. If the limit of infinite spin is taken in such a way that the horizon area per unit length of the ring remains finite, then it approaches a straight, boosted black string. This is also expected to suffer from the GL instability, and therefore a Kerr-bound similar to (7.1) should hold. Note that all known solutions that admit an ultra-spinning regime have some form of black brane limit at infinite rotation. This may be a generic feature.

Finally we would like to conclude with a few comments of the implications of our results for black holes in braneworld scenarios [20]. In these higher-dimensional models, the fundamental scale of gravity is dramatically reduced, possibly even to be O(TeV), and the size of the compact dimensions is increased to be much larger than the corresponding fundamental length scale. Hence one must consider seriously classical solutions of general relativity in higher dimensions. One of the consequences is that there are discernible changes in the properties of black holes when their Schwarzschild radius is less than the compactification scale [21]. In section 3, we showed that the horizon of an ultra-spinning black hole was very extended in the plane of rotation but very narrow in all of the transverse directions. Hence, naively, it would seem that such black holes could arise in braneworld models, and as a consequence, we might expect to find *macroscopic* black holes whose angular momentum far exceeds the standard Kerr bound. An observation of such a black hole would seem to provide clear evidence of extra dimensions. Unfortunately, this line of reasoning is undermined by our discovery that ultra-spinning black holes are unstable. That is, even if such a macroscopic black hole was created, it would rapidly decay and presumably settle down to a configuration which presents no violations of the Kerr bound.

Note, however, that given the inequality (7.1), microscopic black holes need not conform to the four-dimensional Kerr bound. For comparison purposes consider six spacetime dimensions. Then eq. (7.1) yields $J \lesssim (M/M_{\rm fun})^{4/3}$ while the standard Kerr bound becomes $J \lesssim (M/M_{\rm Planck})^2 \simeq M^2/(M_{\rm fun}^4 L_{\rm compact}^2)$. Hence the maximum angular momentum rises more rapidly in the first (higher-dimensional) inequality because of both the power and the prefactor. Of course, the expressions on the l.h.s. of the two inequalities meet roughly when the horizon radius of the black hole reaches the compactification scale $L_{\rm compact}$. Hence the microscopic or higher-dimensional Kerr bound merges smoothly with the standard four-dimensional Kerr bound which still constrains the macroscopic black holes.

Another dramatic prediction of the braneworld scenarios is that it becomes much easier to produce microscopic black holes in subatomic collisions of elementary particles — see, for example, refs. [22]–[24]. In the initial analyses [22], the cross-section for black hole formation in a grazing collision was estimated to be $\pi\, R_S^2$, where R_S was the (higher-dimensional) Schwarzschild radius associated with the center-of-mass energy. In many cases [24] it was suggested that an improvement which took into account the angular momentum associated with particles colliding with a finite impact parameter was to replace R_S with the coordinate radius r_+ determined by eq. (2.5). However, r_+ has no invariant meaning as it can be changed by a coordinate transformation, and moreover, given the discussion of the horizon geometry in section 3, this choice would seem to be a gross underestimate which does not conform with the geometric picture that motivated the original estimate. More convincingly, one can employ analytic and numerical methods [25, 26] to search for an apparent horizon for two particles colliding with a finite impact parameter. This analysis puts an upper bound on the impact parameter in higher dimensions [26].[4]

Hence it appears that the ultra-spinning black holes play no role in the black hole production from particle collisions, and it seems plausible that this result is related to the instability of these solutions. One's usual intuition (as developed in four dimensions) is that black holes are stable objects and hence gravitational collapse inevitably settles down to one of a limited family of black hole solutions. Now roughly one may think of the solutions of Einstein's equations as trajectories in a superspace of spatial geometries, and then the previous intuition could be summarized by saying that the black hole solutions are distinguished as attractors in this space of trajectories. In contrast, the instability of the ultra-spinning black holes suggests that these solutions actually are "repellors" in the space of solutions. That is, nearby trajectories are driven away by the unstable nature of these ultra-spinning solutions. Turning this around then, one might interpret the results on apparent horizons [26] as further evidence that $(a^{d-3}/\mu)_{\rm crit} = O(1)$.

In any event, because of their instability, it seems that ultra-spinning black holes have no dramatic consequences for braneworld scenarios.

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[4]These results have recently been incorporated in the phenomenological analysis of ref. [27].

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A. Black brane limit with several independent spins

We discuss first the case of an odd number of spacetime dimensions d. The solution for a black hole with arbitrary rotation in each of the $(d-1)/2$ independent rotation planes is [1]

$$ds^2 = -dt^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) + \frac{\mu r^2}{\Pi F}(dt + a_i \mu_i^2 d\varphi_i)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 \,. \tag{A.1}$$

Unless otherwise stated, we assume summation over $i = 1, \ldots, \frac{d-1}{2}$. The mass parameter is μ, not to be confused with the direction cosines μ_i, which satisfy $\mu_i^2 = 1$. Here

$$F = 1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \,, \qquad \Pi = \prod_{i=1}^{(d-1)/2} (r^2 + a_i^2) \,. \tag{A.2}$$

We assume that the parameters are such that a horizon exists. A sufficient, but not necessary, condition is that any two of the spin parameters vanish, i.e., if two a_i vanish, a horizon will always exist irrespective of how large the remaining spin parameters are [1].

Take the first n rotation parameters to be comparable among themselves, and much larger than the remaining $(d-1)/2 - n$ ones. We use the index j for the former, and k for the latter. We take the limit

$$\begin{aligned} a_j &\to \infty \,, & j &= 1, \ldots, n \,, \\ a_k &\ \ \text{finite} \,, & k &= n+1, \ldots, \frac{d-1}{2} \,, \end{aligned} \tag{A.3}$$

where the different a_j go to infinity at the same rate. We also take $\mu \to \infty$, keeping

$$\frac{\mu}{\prod_j a_j^2} \equiv \hat{\mu} \tag{A.4}$$

finite, and define new coordinates $\sigma_j = a_j \mu_j$ (no sum) that stay finite as we approach $\mu_j \to 0$. The remaining μ_k must satisfy $\mu_k^2 = 1$. In the limit

$$\begin{aligned} F &\to 1 - \frac{a_k^2 \mu_k^2}{r^2 + a_k^2} \equiv \widehat{F} \,, \\ \Pi &\to \prod_k (r^2 + a_k^2) \prod_j a_j^2 \equiv \widehat{\Pi} \prod_j a_j^2 \,, \end{aligned} \tag{A.5}$$

i.e., $\widehat{F}$ and $\widehat{\Pi}$ are the same functions as in (A.2) but now involving only the $(d-1)/2 - n$ rotation parameters that remain finite. The metric that results is

$$\begin{aligned} ds^2 = &-dt^2 + (r^2 + a_k^2)(d\mu_k^2 + \mu_k^2 d\varphi_k^2) + \frac{\hat{\mu} r^2}{\widehat{\Pi}\widehat{F}}(dt + a_k \mu_k^2 d\varphi_k)^2 + \frac{\widehat{\Pi}\widehat{F}}{\widehat{\Pi} - \hat{\mu} r^2} dr^2 + \\ &+ d\sigma_j^2 + \sigma_j^2 d\varphi_j^2 \,. \end{aligned} \tag{A.6}$$

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The first line is a rotating black hole metric as in (A.1), but in $d-2n$ dimensions. The remaining $2n$ dimensions, in the second line, form a flat $\mathbb{R}^{2n}$ with coordinates (σ_j, φ_j). So the limiting metric is a rotating black $2n$-brane, where the rotation is on the spherical $S^{d-2(n+1)}$ sections of the horizon.

For even d the general solution is

$$ds^2 = -dt^2 + r^2 d\alpha^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) + \frac{\mu r}{\Pi F}(dt + a_i \mu_i^2 d\varphi_i)^2 + \frac{\Pi F}{\Pi - \mu r} dr^2 \,, \quad \text{(A.7)}$$

where now i runs to $d/2$, also in F and Π in (A.2), and $\mu_i^2 + \alpha^2 = 1$. In this case, the existence of a horizon is guaranteed if any one of the spins vanishes. Taking as before n spins to infinity, we find again a rotating black $2n$-brane.

The presence of horizons for generic parameters in (A.1) and (A.7) is difficult to ascertain. If all spin parameters are non-zero, then an upper extremality bound on a combination of the spins arises [1]. If it is exceeded, naked singularities appear, as in the $d = 4$ Kerr black hole. One may worry that, if all spins are turned on, and we send some of them to infinity, then the extremal bound will always be exceeded. This is not the case, since at the same time we are taking $\mu \to \infty$ in an appropriate manner. In the case of even d, sending all parameters but one to infinity results in a Kerr $(d-4)$-brane, and the bound on the spin of the latter is inherited from the higher-dimensional one. Conversely, the equation for the Kerr horizon provides a good approximation to determine the horizon for finite but large values of the spins a_j (i.e., all a_j are large and comparable, and $a_j \gg a_1$):

$$r_+^2 + a_1^2 - \frac{\mu}{\prod_{j>1} a_j^2} r_+ = 0 \,. \quad \text{(A.8)}$$

Note that this matches the equation determining the horizon for the full solution with the approximation $r_+ \ll a_j$. Either approach yields the bound that the minimum value for the mass parameter which yields a regular horizon is given by

$$\mu \gtrsim 2|a_1| \prod_{j>1} a_j^2 \,. \quad \text{(A.9)}$$

In odd d, one can take, at most, all but *two* spins to infinity in order to get a black brane (i.e., there are no black $(d-3)$-branes in d dimensions, since there are no asymptotically flat vacuum black holes in three dimensions). In this case the generic limiting black brane is the product of $\mathbb{R}^{d-5}$ times the doubly-spinning five-dimensional black hole. If both spins are non-zero, then there is a regular extremal limit, while if one of them is zero the extremal solution is singular [1].[5] In each case one can obtain the analogue of eq. (A.8). In general, the condition for the existence of an ultra-spinning regime with n fast spins becomes, in the limit, the same as the condition for the existence of a horizon in $d-2n$ dimensions, with only slow spins, and with mass parameter given by (A.4).

If all the non-infinite rotations a_k are zero, then the analysis of [6] shows that the limiting static black branes are unstable. The results of [6] have not been extended to

[5]It is quite likely that more general higher-dimensional solutions exist which have as a limit the product of the black ring of [4] times a flat $\mathbb{R}^{2n}$.

rotating black branes, not even to the Kerr black branes, and it is not known the range of spins for which an instability occurs. It seems reasonable to expect that for small rotation a_k the instability should be present, but in general a more complicated behavior can be expected, on the basis of a conjectured connection between classical and thermodynamic stability of black branes [11, 13]. This is currently under investigation.

B. Death by radiation

A conservative conjecture as to the evolution of the ultra-spinning black holes is that with the onset of the instability, the black hole horizon will generically be distorted in such a way that its axial symmetry is broken. Since the black hole is rotating, a varying quadrupole moment will appear and gravitational waves will be emitted. In this way, the black hole will shed a fraction of both its spin and its mass. If it loses enough angular momentum, it may return to a range of values of (M, J) within the stable regime. Again, this process must proceed in a way that the horizon area increases monotonically throughout so as to be compatible with Hawking's area theorem.

To produce a precise description of the spin-down above, it seems one can not avoid solving the higher-dimensional Einstein equations in a strong field regime. Hence we were not able to produce a detailed calculation of the gravitational radiation. However, we present some calculations which make some tentative steps in this direction. In particular, we have in mind processes in which the black hole is only slightly perturbed. For example, if the original angular momentum is only slightly beyond the region of stability, one might imagine that throughout the entire evolution the black hole will only deviate from the family of solutions in eq. (2.1). Now as commented above, the area theorem requires that the emission of gravitational radiation must be accompanied by an increase of the horizon area. Having supposed that there are processes whose effects are perturbative, we can examine the variation of the horizon area of the solutions (2.1). Such an analysis might give a hint as to the boundary between the stable and unstable regimes. However, we warn the reader that our results are inconclusive.

We begin with some general observations. Imagine that one detects the radiation at a large sphere surrounding the black hole. In some short interval, it carries away some angular momentum δJ from the black hole, and also some energy δE. On general grounds, one might imagine that δJ is directly proportional to δE. However, on dimensional grounds, we must introduce some physically relevant scale in the proportionality constant. In a standard radiation calculation, this might be the frequency of the emitted waves. In the present case, it turns out to be convenient to use the angular frequency of the horizon,

$$\Omega = \frac{a}{r_+^2 + a^2}\,, \tag{B.1}$$

with which we write

$$\Omega\,\delta J = \alpha\,\delta E\,. \tag{B.2}$$

Here α is a dimensionless coefficient that characterizes the efficiency of the radiation process, i.e., large α corresponds to a process that radiates spin efficiently. Now for a slightly perturbed solution, the first law of black hole mechanics allows us to obtain the variation in the black hole area as

$$\begin{aligned} \frac{\kappa}{8\pi}\,\delta\mathcal{A} &= \Omega\,\delta J - \delta E \\ &= (\alpha - 1)\,\delta E\,, \end{aligned} \tag{B.3}$$

where κ is the surface gravity of the black hole horizon. Hence with the definition of the efficiency factor in eq. (B.2), we have the simple result that demanding that $\delta\mathcal{A} \geq 0$ requires

$$\alpha \geq 1\,. \tag{B.4}$$

This result only predicts that the area theorem can be satisfied for any spin provided that there exist radiation processes with $\alpha > 1$. Of course, we expect that this efficiency can only be attained for large spins, i.e., an instability is only expected to appear in the ultra-spinning regime. To determine what realistic values for efficiency factor might be, we examined two models for the emission of gravitational waves.

The first was to consider the emission from a near-newtonian slow-motion source. In four dimensions, this is a textbook calculation[6] and has been recently been extended to higher dimensions in ref. [29]. We present no details here but rather observe that the final result $\delta E = \omega\,\delta J$ matches the variation in energy and angular momentum loss of the source which rigidly rotates with angular frequency ω (in any spacetime dimension). If we assume that this result were to apply in the present problem with $\omega = \Omega$, it would give $\alpha = 1$ independent of the value of the spin parameter a. The latter is, of course, unexpected since we are looking to find an instability only in the ultra-spinning regime. It is likely that it is simply inappropriate to consider the black hole to be a rigidly rotating body, i.e., , both the angular frequency and the moment of inertia vary during the spin-down.

We also consider a model where the rotating black hole emits a null particle in the plane of rotation. One might think of this process as approximating "radiation" focussed in the equatorial plane. In particular, one can consider the emission of many or several massless particles and the analysis would be essentially unchanged with the contributions of the individual particles combining additively. Unfortunately our final result is that this conjectured decay mechanism does *not* lead to an increase in the area of the horizons, and therefore is ruled out as a possible decay channel. Nevertheless, it provides an instructive example to consider, and it is also an instance where the calculations can be performed quite rigorously, in particular we can determine precisely the efficiency factor α, or equivalently, the impact parameter R of the outgoing particle. Hence we present the calculation in the following subsection.

The latter negative result is somewhat disturbing as since the null particles are confined to the plane of rotation, this would seem the most efficient possible way to dissipate the angular momentum. It is hard to believe that gravitational radiation from distortions in an ultra-spinning horizon would not cause a spin-down of the black hole. We take our present results as an indication that the instability must produce distortions deep in the nonlinear

[6]See, for example, chapter 36 of [28].

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regime for the radiation process to be efficient. That is, we can not rely on calculations which consider only small perturbations of the stationary black hole solutions (2.1).

B.1 Emission of massless particles

Consider that the rotating black hole emits a massless particle from its edge in the plane of rotation, tangentially to the horizon and in the direction of rotation. The particle carries away energy δE and orbital angular momentum δJ, small enough so we can treat it as a test particle in the black hole background. We also consider that the black hole evolves from the solution with mass and spin (M, J) to an infinitesimally close one with $(M - \delta E, J - \delta J)$. The trajectory of the massless particle can be easily computed as a null geodesic in the plane of rotation of the black hole. For a massless particle that at infinity has energy δE and orbital angular momentum δJ, the radial equation is

$$\dot{r}^2 = (\delta E)^2 - V(r) \tag{B.5}$$

with

$$V(r) = -\frac{\mu\,(\delta J - a\,\delta E)^2}{r^{d-1}} + \frac{(\delta J)^2 - a^2\,(\delta E)^2}{r^2}\,, \tag{B.6}$$

and where μ and a give M and J as in (2.3). This potential has a maximum at a critical value

$$r_c = \left(\frac{(d-1)\mu}{2}\frac{\delta J - a\,\delta E}{\delta J + a\,\delta E}\right)^{1/(d-3)}\,. \tag{B.7}$$

A particle going out from $r < r_c$ needs to have enough energy to climb up the gravitational potential in order to escape out to infinity. This determines a minimum energy $(\delta E)^2 = V(r_c)$ and therefore a maximum possible impact parameter $R = \delta J/\delta E$ for a particle escaping the black hole tangentially on the plane of rotation. Then the maximal impact parameter is determined by solving

$$\left(\frac{R}{a}+1\right)^{d-1}\left(\frac{R}{a}-1\right)^{d-5} = \frac{1}{4}\frac{(d-1)^{d-1}}{(d-3)^{d-3}}\left(\frac{\mu}{a^{d-3}}\right)^2\,. \tag{B.8}$$

Now comparing to the previous discussion, we see that the efficiency factor is given by $\alpha = \Omega\,R$. Hence we must determine if this expression can be larger than one.

In the ultra-spinning regime the r.h.s. of eq. (B.8) is a small quantity, and so we must have $R/a \sim 1$.[7] This result agrees with our previous estimate that the geometric size of the pancaked horizon is of order a in the plane of rotation. Solving eq. (B.8) approximately with large a yields

$$\alpha = \frac{a\,R}{r_+^2 + a^2} = 1 + c_d\frac{r_+^2}{a^2} + O\left(\left(\frac{r_+}{a}\right)^3\right) \qquad \text{where} \qquad c_d = \left[\frac{1}{2^{d+1}}\frac{(d-1)^{d-1}}{(d-3)^{d-3}}\right]^{\frac{1}{d-5}} - 1\,. \tag{B.9}$$

Unfortunately, for all dimensions $d \geq 6$, we have $c_d < 0$ and so, even in the extreme ultra-spinning regime, the inequality (B.4) is *not* satisfied. We have also examined the

[7] Here we assume R is positive. $R < 0$ and $R \sim -a$ corresponds to the particle coming in on the other side of the center axis.

solutions to eq. (B.8) numerically for a wide range of d, and in all cases we have found that $\alpha < 1$, i.e., the area never increases by emission of the massless particle.[8] Therefore, this does not seem to be a possible decay channel for ultra-spinning black holes.

Note that the requirement $\alpha > 1$, i.e., $R/a > 1/(a\Omega)$, is more stringent than that in eq. (6.8) which, for large enough a/r_+, only requires $R/a > 2/(d-2)$. This can be attributed to the fact that in the present case one of the decay products (the massless particle) does not contribute to the total final horizon area, in contrast to the fragmentation scenario.

It is interesting to consider this result in a different light by reversing the arrow of time. Doing so, we would be studying the variation in the final area obtained by throwing a massless particle at the rotating black hole, at the maximum possible impact parameter for capture. The analysis above shows now that if the particle is captured by the black hole, then there is an infinitesimally close black hole of mass $M + \delta E$ and spin $J + \delta J$, and with *larger* area, that the system can evolve into. If there had not been any, it would have been a clear sign of an instability — the area theorem forbids a decrease in the area, so a violent back reaction should have been necessary in order for the black hole to evolve into a configuration, of larger area, not infinitesimally close to the initial one.

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